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Complex Differential Systems and Extension of Lyapunov's Method

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1. Many problems concerning the behavior of solutions of differential systems can be made to depend on scalar differential equations. Using a comparison principle of this type and the concept of Lyapunov's function, one of the authors obtained many results of differential systems [1-4].

Bellman [5] has shown that using a vector Lyapunov function, instead of a single Lyapunov function, is advantageous in some situations. This idea was exploited in [6], where the concepts of conditional stability and boundedness of solutions have been defined and sufficient conditions in terms of several Lyapunov functions were obtained.

Assuming the existence of solutions, we introduce in this paper, in a natural way, concepts of stability and boundedness of solutions of a class of complex differential systems. We obtain sufficient conditions in terms of several Lyapunov-like functions such that these concepts hold. Examples are constructed to illustrate our results.

2. Let D denote the region of the complex plane $|z| \geq a$ and $\alpha \leq \arg z \leq \beta$ where a, α , and β are real numbers. Let C^n denote the n -dimensional complex Euclidian space.

We consider the differential system

$$y' = f(z, y), \quad y(z_0) = y_0, \quad (z_0 \in D), \quad (2.1)$$

where y and f are n -dimensional vectors and the function $f(z, y)$ is defined, regular-analytic in z on D and entire in y on C^n .

By a solution of (2.1), we mean a vector-function $y = y(z)$, with $y(z_0) = y_0$, ($z_0 \in D$), that is regular-analytic in z on D and satisfies (2.1) for all $z \in D$.

In what follows we shall denote $|z|$ by t and $\arg z$ by θ .

Let $y(z)$ be any solution of (2.1). Define $\|y(z)\| = \sum_{i=1}^n |y_i(z)|$. In order to unify our results on stability and boundedness we list below the following conditions.

(i) For each $\epsilon > 0$ and $z_0 \in D$, there exists a positive function $\delta = \delta(|z_0|, \epsilon)$ that is continuous in $|z_0|$ for each ϵ , such that $\|y(z)\| < \epsilon$ for all $|z| \geq |z_0|$, $z \in D$, whenever $\|y(z_0)\| \leq \delta$.

(ii) The δ in (i) is independent of $|z_0|$.

(iii) For each $\gamma \geq 0$ and $z_0 \in D$, there is a positive function $\eta = \eta(|z_0|, \gamma)$ that is continuous in $|z_0|$ for each γ , such that $\|y(z)\| < \eta$ for all $|z| \geq |z_0|$, $z \in D$, whenever $\|y(z_0)\| \leq \gamma$.

(iv) The η in (iii) is independent of $|z_0|$.

(v) For each $\epsilon > 0$, $\gamma \geq 0$ and $z_0 \in D$, there is a positive number $T = T(|z_0|, \epsilon, \gamma)$ such that $\|y(z)\| < \epsilon$ for all $z \in D$, ($|z| \geq |z_0| + T$), whenever $\|y(z_0)\| \leq \gamma$.

(vi) The T in (v) is independent of $|z_0|$.

(vii) For each $\gamma \geq 0$ and $z_0 \in D$, there is a positive number B and a positive number $T = T(|z_0|, \gamma)$, such that $\|y(z)\| < B$ for $z \in D$, ($|z| \geq |z_0| + T$), whenever $\|y(z_0)\| \leq \gamma$.

(viii) The T in (vii) is independent of $|z_0|$.

REMARKS. The definitions given above are natural extensions of the usual definitions of stability and boundedness of solutions of ordinary differential systems, with respect to the origin. We have not assumed that the origin is an invariant set for the system (2.1). Such a possibility, whenever it exists, is implied by the definitions. For example, if the solutions of (2.1) satisfy (i) (since $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$), it is evident that the origin is an invariant set of the system. On the other hand, this need not be, in general, the case with respect to the conditions (iii) to (viii), since we have not assumed the uniqueness of solutions.

3. Let I denote the interval $0 \leq a \leq t < \infty$. Let R^n be the n -dimensional Euclidian-space. We shall say that a function $W(t, r)$ has the property (I) if W and r are n -dimensional vectors, $W(t, r)$ is defined and continuous on the product space $I \times R^n$ and is such, that for each i , $W_i(t, r_1, r_2, \dots, r_n)$ is nondecreasing in $r_1, r_2, \dots, r_{i-1}, r_{i+1}, \dots, r_n$ for each $t \in I$.

Assume that $W(t, r)$ has the property (I). It is known [7] that the differential system

$$r' = W(t, r), \quad r(t_0) = r_0, \quad (3.1)$$

has the maximal solution (in the sense of component wise majorization) existing to the right of t_0 .

4. Let a function $V(z, y)$, where V and y are n -dimensional vectors, be defined, regular-analytic in z on D and entire in y on C^n . The i th component

of V will be denoted by $V_i(z, y)$ or synonymously by $V(z, y_1, y_2, \dots, y_n)$ whenever necessary. Define:

$$V^*(z, y) = \frac{\partial V}{\partial z} + \frac{\partial V}{\partial y} \cdot f(z, y), \quad (4.1)$$

where \cdot denotes scalar multiplication.

With respect to the functions defined above we state the following two lemmas.

LEMMA 1. *Let the function $V^*(z, y)$ of (4.1) satisfy for each i the inequality*

$$|V_i^*(z, y)| \leq W_i(|z|, |V_1(z, y)|, \dots, |V_n(z, y)|), \quad (4.2)$$

where $W(t, r)$ has the property (I). Suppose $r(t)$ is the maximal solution of (3.1). Let $y(z)$ be any solution of (2.1) such that, for each i ,

$$|V_i(z_0, y(z_0))| \leq r_i(t_0), \quad (|z_0| = t_0).$$

Then for each i ,

$$|V_i(z, y(z))| \leq r_i(t), \quad (z \in D, |z| = t), \quad (4.3)$$

for all $t \geq t_0$.

LEMMA 2. *Suppose that $p(z)$ is defined and regular-analytic in z on D . Let the inequality (4.2) be replaced by*

$$\begin{aligned} &|V_i^*(z, y)p(z) + V_i(z, y)p'(z)| \\ &\leq W_i(|z|, |V_1(z, y)p(z)|, \dots, |V_n(z, y)p(z)|) \end{aligned} \quad (4.4)$$

where $W(t, r)$ has the property (I). Suppose $r(t)$ is the maximal solution of (3.1). If $y(z)$ is any solution of (2.1) such that, for each i ,

$$|V_i(z_0, y(z_0))p(z_0)| \leq r_i(t_0), \quad (|z_0| = t_0),$$

then, for each i ,

$$|V_i(z, y(z))p(z)| \leq r_i(t), \quad (z \in D, |z| = t), \quad (4.5)$$

for all $t \geq t_0$.

REMARK. It will be easily seen that Lemma 1 can be deduced from Lemma 2 by taking $p(z) \equiv 1$. We have stated it separately since it is a basic result in the proof of some of the theorems that follows. We only prove the Lemma 2.

PROOF. Define $L_i(z, y(z)) = V_i(z, y(z))p(z)$. Let $y(z)$ be any solution of (2.1) such that $|L_i(z_0, y(z_0))| \leq r_i(t_0)$. For each fixed θ , let $R_i(t, \cdot) = |L_i(z, y(z))| = L_i(te^{i\theta}, y(te^{i\theta}))$. For small $h \geq 0$, we have

$$R_i(t+h, \cdot) - R_i(t, \cdot) \leq |L_i(t + he^{i\theta}, y(t + he^{i\theta})) - L_i(te^{i\theta}, y(te^{i\theta}))| \quad (4.6)$$

It can be easily verified that

$$\left| \frac{\partial R_i(t, \cdot)}{\partial t} \right| \leq \left| \frac{\partial L_i(z, y(z))}{\partial t} \right|.$$

But

$$\left| \frac{\partial L_i(z, y(z))}{\partial t} \right| = \left| \frac{dL_i(z, y(z))e^{i\theta}}{dz} \right| = \left| \frac{dL_i(z, y(z))}{dz} \right|.$$

Therefore,

$$\left| \frac{\partial R_i(t, \cdot)}{\partial t} \right| \leq \left| \frac{dL_i(z, y(z))}{dz} \right|. \quad (4.7)$$

It follows, therefore, from (4.7), (4.6), and (4.4) and the definition of $L_i(z, y(z))$ that

$$\limsup_{h \rightarrow 0^+} \frac{R_i(t+h, \cdot) - R_i(t, \cdot)}{h} \leq W_i(|z|, R_1(t, \cdot), R_2(t, \cdot), \dots, R_n(t, \cdot))$$

From the monotonic property of W and by following the arguments in [7], we can establish the Lemma.

5. Corresponding to the conditions (i) to (iv) given above in Section 2, we say that the differential system (3.1) has the property (ia) whenever the following condition is satisfied. (ia) Given $\epsilon > 0$ and $t_0 \in I$, there is a positive function $d = d(t_0, \epsilon)$ that is continuous in t_0 , for each ϵ , such that $\sum_{i=1}^n r_i(t) < \epsilon$, for $t \geq t_0$ whenever $\sum_{i=1}^n r_i(t_0) \leq d$. We can similarly formulate the conditions (iia), (iiaa), and (iva). Let $\phi(t) = \min_{\theta, \theta \in D} |p(z)|$, where $p(z)$ is the same function as defined in Lemma 2. Corresponding to the conditions (v) to (viii) in the same section, we say that the differential system (3.1) has the property (va) if the following condition is satisfied.

(va) Given $\epsilon > 0$, $\gamma \geq 0$, and $t_0 \in I$, there is a positive number $T = T(t_0, \epsilon, \gamma)$ such that

$$\sum_{i=1}^n r_i(t) < \epsilon \phi(t) \quad \text{for all } t \geq t_0 + T \quad \text{whenever} \quad \sum_{i=1}^n r_i(t_0) \leq V \phi(t_0).$$

We can similarly formulate the conditions (via) to (viiia). Assume that

(5.1) The function $b(r)$ is continuous and nondecreasing in r , $b(r) > 0$ for $r > 0$ and $b(\|y\|) \leq \|V(z, y)\|$ for all $z \in D$,

$$\|V(z, y)\| \rightarrow 0 \text{ as } \|y\| \rightarrow 0 \text{ for each } z \in D, \quad (5.2)$$

$$\|V(z, y)\| \rightarrow 0 \text{ as } \|y\| \rightarrow 0 \text{ uniformly in } z \text{ for } z \in D, \quad (5.3)$$

and

$$b(r) \rightarrow \infty \text{ as } r \rightarrow \infty. \quad (5.4)$$

We have the following theorems on stability and boundedness of solutions of (2.1).

THEOREM 1. *Let the assumptions of Lemma 1 hold together with (5.1) and (5.2). Then if the condition (ia) holds, the corresponding condition (i) holds. If (5.2) is strengthened to (5.3) and (iia) holds, then the corresponding condition (ii) holds.*

PROOF. For any $\epsilon > 0$, if $\|y\| \geq \epsilon$, we get from (5.1) that

$$b(\epsilon) \leq b(\|y\|) \leq \|V(z, y)\| \quad \text{for all } z \in D. \quad (5.5)$$

If (ia) holds, given $b(\epsilon) > 0$ and $t_0 \in I$, there is a positive function $d = d(t_0, \epsilon)$ that is continuous in t_0 for each ϵ , such that

$$\sum_{i=1}^n r_i(t) < b(\epsilon) \quad \text{for all } t \geq t_0, \quad (5.6)$$

whenever

$$\sum_{i=1}^n r_i(t_0) \leq d. \quad (5.7)$$

In view of (5.2) there is a $\delta = \delta(\|z_0\|, \epsilon)$ such that

$$\sup_{\|y(z_0)\| \leq \delta} \|V(z_0, y(z_0))\| \leq d. \quad (5.8)$$

As $\sum_{i=1}^n |V_i(z, y(z))| = \|V(z, y(z))\|$, it follows from Lemma 1 that

$$\|V(z, y(z))\| \leq \sum_{i=1}^n r_i(t), \quad \text{for all } \|z\| = t \geq t_0, \quad (5.9)$$

whenever

$$\|V(z_0, y(z_0))\| \leq \sum_{i=1}^n r_i(t_0). \quad (5.10)$$

Now choose

$$\sum_{i=1}^n r_i(t_0) \leq d.$$

Then it follows from (5.8) and (5.10) that every solution $y(z)$ of (2.1) satisfies (5.9) whenever $\|y(z_0)\| \leq \delta$. Suppose for some $|z_1| > |z_0|$, $z_1 \in D$, a solution $y(z)$, with $\|y(z_0)\| \leq \delta$, is such that $\|y(z_1)\| = \epsilon$. Then using the relations (5.5), (5.6), and (5.9), we have

$$b(\epsilon) \leq \|V(z_1, y(z_1))\| \leq \sum_{i=1}^n r_i(t_1) < b(\epsilon),$$

a contradiction, which proves the first part of the theorem. The second part of the theorem follows easily as d and δ will be independent of t_0 .

THEOREM 2. *Let the assumptions of Lemma 1 hold together with the conditions (5.1), (5.2), and (5.4). Then, if the condition (iiia) holds, the corresponding condition (iiia) holds. If (5.2) is strengthened to (5.3) and condition (iva) holds, the corresponding condition (iv) also holds.*

PROOF. Let $\gamma \geq 0$ and $z_0 \in D$ be given. Let $\|y(z_0)\| \leq \gamma$. Then, in view of the condition (5.2), there is a $\gamma_1 = \gamma_1(|z_0|, \gamma)$ that is continuous in $|z_0|$ for each γ , such that

$$\sup_{\|y(z_0)\| \leq \gamma} \|V(z_0, y(z_0))\| \leq \gamma_1. \quad (5.11)$$

Since the condition (iiia) holds, given $\gamma_1 \geq 0$ and $t_0 \in I$ there is a positive function $\eta = \eta(t_0, \gamma)$ that is continuous in t_0 for each γ , such that

$$\sum_{i=1}^n r_i(t) < \eta, \quad \text{for all } t \geq t_0 \quad (5.12)$$

whenever

$$\sum_{i=1}^n r_i(t_0) \leq \gamma_1.$$

Choosing $\sum_{i=1}^n r_i(t_0) \leq \gamma_1$ and proceeding as in Theorem 1, we conclude that whenever $\|y(z_0)\| \leq \gamma$, every solution $y(z)$ of (2.1) satisfies (5.9).

As (5.4) is satisfied, there is a $L = L(t_0, \gamma)$ such that

$$\eta \leq b(L). \quad (5.13)$$

Consider a solution $y(z)$ of (2.1), with $\|y(z_0)\| \leq \gamma_1$. Let $z_1 \in D$ and $|z_1| > |z_0|$. Suppose that z_1 is such that $\|y(z_1)\| = L$. This, in view of the relations (5.1), (5.9), (5.12) and (5.13), leads to the contradiction

$$b(L) \leq b(\|y(z_1)\|) \leq \|V(z_1, y(z_1))\| \leq \sum_{i=1}^n r_i(t_1) < \eta \leq b(L).$$

The proof of the first part of the theorem is completed. The second part follows easily as γ_1 and L will be, in this case, independent of $|z_0| = t_0$.

THEOREM 3. *Let the assumptions of Lemma 2 hold together with (5.1) and (5.2). Then, if the condition (va) holds, the corresponding condition (v) holds. If (5.2) is strengthened to (5.3) and condition (via) holds, then the corresponding condition (vi) holds.*

PROOF. Let $\epsilon > 0$, $\gamma \geq 0$, and $z_0 \in D$ be given. Suppose $\|y(z_0)\| \leq \gamma$ and γ_1 is the function as defined in (5.11).

As the condition (va) holds, given $b(\epsilon) > 0$, $\gamma_1 \geq 0$ and $t_0 \in I$, there is a positive function $\phi(t)$ and a positive number $T = T(t_0, \epsilon, \gamma)$ such that

$$\sum_{i=1}^n r_i(t) < b(\epsilon)\phi(t) \quad \text{for all } t \geq t_0 + T, \quad (5.14)$$

whenever

$$\sum_{i=1}^n r_i(t_0) \leq \gamma_1 \phi(t_0). \quad (5.15)$$

In view of Lemma 2, we have

$$\|V(z, y(z))\| \phi(t) \leq \sum_{i=1}^n r_i(t), \quad \text{for all } |z| = t \geq t_0, \quad (5.16)$$

whenever

$$\|V(z_0, y(z_0))\| \phi(t_0) \leq \sum_{i=1}^n r_i(t_0). \quad (5.17)$$

Now choose $r_i(t_0)$, so that $\sum_{i=1}^n r_i(t_0) \leq \gamma_1 \phi(t_0)$. It follows from (5.17) and (5.11) that whenever $\|y(z_0)\| \leq \gamma$, every solution $y(z)$ of (2.1) satisfies (5.16).

Let $y(z)$ be any solution of (2.1), with $\|y(z_0)\| \leq \gamma$. Choose a sequence $\{z_n\}$, $z_n \in D$, where $|z_n| = t_n \geq t_0 + T$ and $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Suppose

that $\|y(z_n)\| \geq \epsilon$. In view of the relations (5.5), (5.16), and (5.14) this leads to the contradiction

$$b(\epsilon) \leq \|V(z_n, y(z_n))\| \leq \frac{\sum_{i=1}^n r_i(t_n)}{\phi(t_n)} < b(\epsilon),$$

proving the first part of the theorem. The second part follows immediately since, due to the assumptions in this case γ_1 and T will be independent of $t_0 = |z_0|$.

THEOREM 4. *Let the assumptions of Lemma 2 hold together with (5.1), (5.2), and (5.4). Then, if the condition (viiia) holds, the corresponding condition (vii) holds. If (5.2) is strengthened to (5.3) and condition (viiia) holds, then the corresponding condition (viii) holds.*

PROOF. Let $\gamma \geq 0$ and $z_0 \in D$ be given. Let $\|y(z_0)\| \leq \gamma$ and γ_1 be the same function as defined in (5.11).

As the condition (viiia) holds given $\gamma_1 \geq 0$, $t_0 \in I$, there exists a positive number η and $T = T(t_0, \gamma)$ that is continuous in t_0 for each γ and a function $\phi(t)$ which is continuous in t for $t \geq t_0$, such that

$$\sum_{i=1}^n r_i(t) < \eta\phi(t) \quad \text{for all } t \geq t_0 + T, \quad (5.18)$$

whenever

$$\sum_{i=1}^n r_i(t_0) \leq \gamma_1\phi(t_0).$$

Now choose $r_i(t_0)$ so that $\sum_{i=1}^n r_i(t_0) \leq \gamma_1\phi(t_0)$. Then, proceeding as in Theorem 3, we conclude that any solution $y(z)$ of (2.1), for which $\|y(z_0)\| \leq \gamma$, satisfies (5.16).

As (5.4) holds, there is a L such that

$$\eta \leq b(L). \quad (5.19)$$

Consider a solution $y(z)$ of (2.1), with $\|y(z_0)\| \leq \gamma$. Let $\{z_n\}$ be a sequence such that $z_n \in D$, $|z_n| = t_n \geq t_0 + T$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and $\|y(z_n)\| \geq L$. This, due to the relations (5.1), (5.16), (5.18), and (5.19) leads to the contradiction

$$b(L) \leq \|V(z_n, y(z_n))\| \leq \frac{\sum_{i=1}^n r_i(t_n)}{\phi(t_n)} < \eta \leq b(L).$$

Thus the first part of the theorem is proved. The second part follows easily.

Note: If the conditions of Theorems 1 and 3 hold simultaneously we have equiasymptotic stability for the system (2.1). Similarly other combinations could be deduced.

We add below a few examples to illustrate the results.

Example 1. Let D denote the region of the complex plane $|z| > 0$ and $-\pi/2 \leq \arg z \leq +\pi/2$.

Consider the differential system

$$\begin{aligned} y_1' &= \frac{1}{z^3} [(4-z)y_1 + 2(1-z)y_2] \\ y_2' &= \frac{1}{z^3} [2(z-1)y_1 + (-1+4z)y_2] \end{aligned} \quad (6.1)$$

Let

$$\begin{aligned} V_1(z, y_1, y_2) &= \frac{(2y_1 + y_2)}{3} \exp\left(\frac{3}{2z^2}\right) \\ V_2(z, y_1, y_2) &= \frac{(y_1 + 2y_2)}{3}. \end{aligned} \quad (6.2)$$

As $|\exp(3/2z^2)| > 1$ for all $z \in D$, taking $b(r) = r/3$, we observe that all the assumptions of (5.1) are satisfied. (5.2) is obviously satisfied.

From (4.1) and (6.2) we deduce that $|V_1^*(z, y_1, y_2)| = 0$ and

$$|V_2^*(z, y_1, y_2)| = |V_2|/|z^2|.$$

Hence the differential system (3.1) reduces to

$$\begin{aligned} r_1' &= 0 \\ r' &= r_2/t^2, \end{aligned} \quad (6.3)$$

whose solution is given by

$$\begin{aligned} r_1(t) &= r_1(t_0), \\ r_2(t) &= r_2(t_0) \exp\left[-\frac{1}{t} + \frac{1}{t_0}\right]. \end{aligned}$$

It is obvious that condition (ia) is satisfied by (6.3). Hence by Theorem 1, system (6.1) has the property (i).

Example 2. Let D be the same domain of the complex plane as defined in Example 1.

Consider the differential equation

$$y' = -\left(1 + \frac{1}{z}\right)y. \quad (6.4)$$

Taking $V(z, y) = y^2$, we find that condition (5.1) is satisfied if $b(r) = r^2$. Choose $p(z) = z^3 \exp(2z)$. Then Eq. (3.1) reduces to

$$r' = r/t, \quad (6.5)$$

whose solution is $r(t) = (r(t_0)/t_0)t$. Observing that $\phi(t) = t^3$, it is easily seen that condition (va) is satisfied by (6.5).

Hence by Theorem 3, (6.4) has the property (v).

Example 3: Let D be the domain of the complex plane $|z| > 1$,

$$0 \leq \arg z \leq 2\pi$$

Consider the differential equation

$$y' = -\frac{2z}{1+z^2}y + \frac{1}{z^2(1+z^2)}. \quad (6.6)$$

Let $V(z, y) = y$ and $p(z) = (1+z^2)$. Taking $b(r) = r$, all the assumptions of (5.1) and (5.2) are satisfied.

We have

$$p(z)V^*(z, y) + V(z, y)p'(z) = \frac{1}{z^2}.$$

Hence the differential equation (3.1) reduces to

$$r' = 1/t^2, \quad (6.7)$$

whose solution is given by

$$r(t) = -\frac{1}{t} + r(t_0) + \frac{1}{t_0}.$$

Observing that $\phi(t) = t^2 - 1$, it can be easily seen that (6.7) has the property (va). Hence by Theorem 3, (6.6) has the property (v).

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